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Bivariate Bernstein-Szegő polynomials

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(joint work with Jeffrey S. Geronimo)

The Bernstein-Szegő measures and polynomials on the real line play an important role in probability, numerical analysis and approximation theory. In this talk, I will define bivariate extensions of these measures and associated spaces of polynomials, and discuss their spectral and characteristic properties. The talk is based on the work [4].

Bernstein-Szegő measures on $\mathbb{R}$. An important class of measures on $\mathbb{R}$ introduced by Bernstein and Szegő are the measures of the form

$$d\mu = \frac{2}{\pi}\frac{\sqrt{1-x^2}}{Q(x)}\chi_{(-1,1)}(x)\,dx, \tag{1}$$

where $Q(x)$ is a polynomial nonvanishing on $(-1,1)$, with at most simple zeros at $x = \pm 1$ and χ_J denotes the characteristic function of a set J. Recall that if $\{p_k(x)\}_{k=0}^{\infty}$ are orthonormal polynomials with respect to a measure μ on the real line, then the multiplication by x can be represented by a three-term operator

$$a_{k+1}p_{k+1}(x) + b_k p_k(x) + a_k p_{k-1}(x) = xp_k(x).$$

Suppose that $Q(x)$ is a polynomial of degree at most $2n$ for some positive integer n, and let $q(z)$ denote the stable Fejér-Riesz factor of $Q(x)$, i.e. $q(z)$ is the unique polynomial with real coefficients and no zeros in the closed unit disk, except possibly for simple zeros at $z = \pm 1$, such that

$$Q(x) = q(z)q(1/z), \quad \text{where} \quad x = \frac{1}{2}\left(z + \frac{1}{z}\right),$$

normalized so that $q(0) > 0$. We can define orthonormal polynomials with respect to μ in (1) by

$$p_k(x) = \frac{z^{k+1}q(1/z) - z^{-k-1}q(z)}{z - 1/z} \quad \text{for } k \geq n. \tag{2}$$

The last equation implies that

$$a_{k+1} = \frac{1}{2} \quad \text{and} \quad b_k = 0 \qquad \text{for } k \geq n. \tag{3}$$

Conversely, if (3) holds then (1) holds for some polynomial $Q(x)$ of degree at most $2n$ if and only if

$$q(z) = z^n \left(p_n(x) - 2za_n p_{n-1}(x)\right) \neq 0 \qquad \text{for} \qquad z \in (-1,1),$$

see [1, 2] and the references therein.

Bivariate extension. Let $\mathbb{R}[x,y]$ denote the space of all polynomials of x and y with real coefficients, and for $k,l \in \mathbb{N}_0$, let $\mathbb{R}_k[x] = \text{span}_{\mathbb{R}}\{x^i : 0 \leq i \leq k\}$, $\mathbb{R}_l[y] = \text{span}_{\mathbb{R}}\{y^i : 0 \leq i \leq l\}$, $\mathbb{R}_{k,l}[x,y] = \text{span}_{\mathbb{R}}\{x^i y^j : 0 \leq i \leq k, 0 \leq j \leq l\}$. For a measure μ on $\mathbb{R}^2$ we set

$$\mathsf{P}_{k,l;\mu}[x,y] = \mathbb{R}_{k,l}[x,y] \ominus \mathbb{R}_{k-1,l}[x,y] \text{ and } \tilde{\mathsf{P}}_{k,l;\mu}[x,y] = \mathbb{R}_{k,l}[x,y] \ominus \mathbb{R}_{k,l-1}[x,y].$$

We can construct an orthonormal basis $\{p^j_{k,l}(x,y) : 0 \leq j \leq l\}$ of the space $\mathsf{P}_{k,l;\mu}[x,y]$ using lexicographical order of the monomials and we set $P_{k,l}(x,y) = [p^0_{k,l}(x,y), p^1_{k,l}(x,y), \dots, p^l_{k,l}(x,y)]^t$. Similarly, we use reverse lexicographical order of the monomials to construct a basis $\{\tilde{p}^j_{k,l}(x,y) : 0 \leq j \leq k\}$ for $\tilde{\mathsf{P}}_{k,l;\mu}[x,y]$, and we set $\tilde{P}_{k,l}(x,y) = [\tilde{p}^0_{k,l}(x,y), \tilde{p}^1_{k,l}(x,y), \dots, \tilde{p}^k_{k,l}(x,y)]^t$. These vector polynomials satisfy the following recurrence relations

$$xP_{k,l}(x,y) = A_{k+1,l}P_{k+1,l}(x,y) + B_{k,l}P_{k,l}(x,y) + A^t_{k,l}P_{k-1,l}(x,y),$$
$$y\tilde{P}_{k,l}(x,y) = \tilde{A}_{k,l+1}\tilde{P}_{k,l+1}(x,y) + \tilde{B}_{k,l}\tilde{P}_{k,l}(x,y) + \tilde{A}^t_{k,l}\tilde{P}_{k,l-1}(x,y),$$

where $A_{k,l}, B_{k,l}$ are $(l+1)\times(l+1)$ matrices and $\tilde{A}_{k,l}, \tilde{B}_{k,l}$ are $(k+1)\times(k+1)$ matrices. Suppose that $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$, $y = \frac{1}{2}\left(w + \frac{1}{w}\right)$ and

- $\omega(z,w) \in \mathbb{R}_{n_0,m_0}[z,w]$ is nonzero for $|z| \leq 1$, $|w| \leq 1$;
- $q_1(x) \in \mathbb{R}_{2n_1}[x]$ is positive for $x \in (-1,1)$, having at most simple zeros at ± 1;
- $q_2(y) \in \mathbb{R}_{2m_1}[y]$ is positive for $y \in (-1,1)$, having at most simple zeros at ± 1.

Then the recurrence coefficients of the measure

$$d\mu(x,y) = \frac{4}{\pi^2}\frac{\chi_{(-1,1)^2}(x,y)\sqrt{1-x^2}\sqrt{1-y^2}}{q_1(x)q_2(y)\omega(z,w)\omega(1/z,w)\omega(z,1/w)\omega(1/z,1/w)}\,dx\,dy, \tag{4}$$

satisfy

$$A_{k+1,l} = \frac{1}{2}I_{l+1}, \qquad B_{k,l} = 0, \qquad \text{for all } k \geq n, \quad l \geq m, \tag{5a}$$

$$\tilde{A}_{k,l+1} = \frac{1}{2}I_{k+1}, \qquad \tilde{B}_{k,l} = 0, \qquad \text{for all k} \geq n, \quad l \geq m, \tag{5b}$$

where $n = n_0 + n_1$, $m = m_0 + m_1$. In view of the spectral properties (5), we can regard the measures (4) as bivariate extensions of the Bernstein-Szegő measures. Note that (5) are invariant if we replace $P_{k,l}(x,y)$ by $O_l P_{k,l}(x,y)$ and $\tilde{P}_{k,l}(x,y)$ by $\tilde{O}_k\tilde{P}_{k,l}(x,y)$, where O_l and $\tilde{O}_k$ are orthogonal matrices depending only on l and k, respectively. We use this freedom to define explicit bases of the spaces $\mathsf{P}_{k,l;\mu}[x,y]$, $\tilde{\mathsf{P}}_{k,l;\mu}[x,y]$ for $k \geq n$ and $l \geq m$ which provide a bivariate extension of the Szegő mapping (2). Let $\tilde{q}_1(z)$ and $\tilde{q}_2(w)$ be the stable Fejér-Riesz factors of $q_1(x)$ and $q_2(y)$, respectively. If $\{U_j^{q_2}(y)\}$ denote the orthonormal polynomials with respect to $\frac{2}{\pi}\frac{\sqrt{1-y^2}}{q_2(y)}\chi_{(-1,1)}(y)dy$ on the real line, and if we set

$$\hat{p}(z,y) = \tilde{q}_1(z)\omega(z,w)\omega(z,1/w) \text{ and } \hat{p}_k(x;y) = \frac{z^{k+1}\hat{p}(1/z,y) - z^{-k-1}\hat{p}(z,y)}{z-1/z},$$

then the one-dimensional Bernstein-Szegő theory implies that $\{\hat{p}_k(x;y)U_j^{q_2}(y)\}_{j=0}^{l-m_0}$ are orthonormal elements in $\mathsf{P}_{k,l;\mu}[x,y]$. Note that we have already used the stable Fejér-Riesz factor of the inverse of the weight, and we need m_0 new quantities for a basis of the complement. It turns out that the elements in the space $\mathsf{P}_{n_0,m_0-1;\mu_\omega}[z,w] = \mathbb{R}_{n_0,m_0-1}[z,w] \ominus \mathbb{R}_{n_0-1,m_0-1}[z,w]$ for the Bernstein-Szegő measure $d\mu_\omega = \frac{1}{(2\pi)^2}\frac{|dz|\,|dw|}{|\omega(z,w)|^2}$ on the torus $\mathbb{T}^2 = \{(z,w) \in \mathbb{C}^2 : |z| = |w| = 1\}$ can be used to build the necessary m_0 orthonormal elements in the complement. On the space of Laurent polynomials $\mathbb{R}[z^{\pm 1}, w^{\pm 1}]$ we define the involution $\mathcal{R}_w^{n_0}$ by $\mathcal{R}_z^{n_0}(g(z,w)) = z^{n_0}g(1/z,w)$, and for $f(z,w) \in \mathbb{R}[z,w]$ we denote by $\mathcal{M}_{f(z,w)}$ the multiplication by $f(z,w)$, i.e. $\mathcal{M}_{f(z,w)}(g(z,w)) = f(z,w)g(z,w)$. Finally, let $S_{z,k}$ and $S_{w,l}$ denote the mappings $S_{z,k}(f(z)) = \frac{z^{k+1}f(1/z)-z^{-k-1}f(z)}{z-1/z}$, $S_{w,l}(g(w)) = \frac{w^{l+1}g(1/w)-w^{-l-1}g(w)}{w-1/w}$, and $\mathcal{S}_{k,l} = S_{z,k} \circ S_{w,l} : \mathbb{R}[z,w] \to \mathbb{R}[x,y]$. With these notations, we define $\mathcal{T}_{k,l} = \mathcal{S}_{k,l} \circ \mathcal{M}_{\tilde{q}_1(z)\tilde{q}_2(w)\omega(z,w)} \circ \mathcal{R}_z^{n_0}$.
Then $\mathcal{T}_{k,l} : \mathsf{P}_{n_0,m_0-1;\mu_\omega}[z,w] \to \mathsf{P}_{k,l;\mu}[x,y]$ is an isometry and

$$\mathsf{P}_{k,l;\mu}[x,y] = \mathcal{T}_{k,l}(\mathsf{P}_{n_0,m_0-1;\mu_\omega}[z,w]) \bigoplus_{j=0}^{l-m_0} \mathrm{span}_{\mathbb{R}}\{\hat{p}_k(x;y)U_j^{q_2}(y)\}.$$

Note that the space $\mathsf{P}_{n_0,m_0-1;\mu_\omega}[z,w]$ in the last formula is independent of k and l. Therefore, if we fix an orthonormal basis of this space, the multiplications by x and y on its image in $\mathsf{P}_{k,l;\mu}[x,y]$ will be represented by Chebyshev relations. An analogous decomposition holds for $\tilde{\mathsf{P}}_{k,l;\mu}[x,y]$ and can be obtained by exchanging the roles of x and y.

Detailed proofs, examples, different extensions of the above constructions and connections to the theory of matrix-valued orthogonal polynomials can be found in [4]. In particular, an interesting new phenomenon in the bivariate case is that the characterization of the Bernstein-Szegő measures in terms of finitely many moments requires new polynomial identities which connect the Fejér-Riesz factorizations of the weight (4) to canonical polynomials depending on three variables associated with a measure on $\mathbb{R}^2$. A challenging open question is to give a complete characterization of the Bernstein-Szegő measures in terms of appropriate recurrence coefficients for the orthogonal spaces similarly to the spectral characterization of the Bernstein-Szegő measures on the torus $\mathbb{T}^2$ in [3]. Another interesting direction is to explore applications of the bivariate Bernstein-Szegő polynomials in numerical analysis, approximation theory and probability.

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Sobolev orthogonal polynomials and spectral methods in boundary value problems

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(joint work with Lidia Fernández, Francisco Marcellán, and Teresa E. Pérez)

The solution of the boundary value problem (BVP, in short) for the ordinary linear differential equation associated with a stationary Schrödinger equation with potential $V(x) = x^{2k}$,

$$\begin{aligned} -u'' + \lambda\, x^{2k}\, u &= f(x), \\ u(-1) = u(1) &= 0, \end{aligned} \tag{1}$$

where $\lambda > 0$, can be studied from a variational perspective according to the fact you can associate a Sobolev inner product

$$\langle u, v\rangle_\lambda = \lambda \int_{-1}^{1} u(x)\, v(x)\, x^{2k}\, dx + \int_{-1}^{1} u'(x)\, v'(x)\, dx, \tag{2}$$

appearing in the variational formulation of (1). This problem, when $k = 1$, has been considered in [5].

Orthogonal polynomials with respect to Sobolev inner products

$$\langle f, g\rangle_S = \int f(x)g(x)d\mu_0(x) + \int f'(x)g'(x)d\mu_0(x), \tag{3}$$

defined by a pair of positive measures (μ_0, μ_1) supported on the real line have attracted the interest of many researchers (see [8], [9], and references therein). They are interesting from several points of view. In approximation theory they constitute a basic tool in smooth approximations by polynomials in the framework of least square problems (see the seminal paper [7]). On the other hand, some authors have considered Fourier expansions in terms of those polynomials as an alternative to the standard ones (see [6]). In numerical analysis, for spectral methods for boundary value problems for ordinary differential equations the Sobolev orthogonal polynomials play an efficient role with respect to the classical ones (see [2], [3], [4]). Indeed, they have been recently studied in the framework of the so called diagonalized spectral methods for boundary value problems for some elliptic differential operators, see [1], [10].